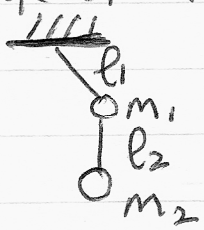
# Linear Chains, etc.

→ Small Oscillations II — { Chains, Strings and the Transition Discrete → Continuous

– previously considered few-degree-of-freedom systems, i.e.

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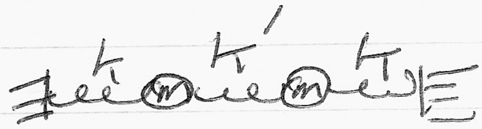
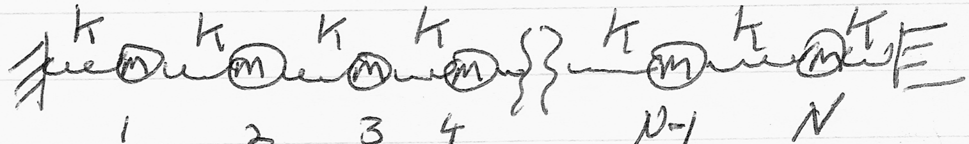



now, consider systems with $N \gg 1$ degrees of freedom, i.e.

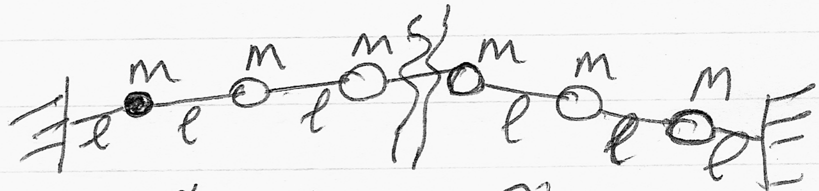
(separated by $\ell$ at equilibrium)

i) – linear chain (1D oscillators) (application → solid)




(identical components) → monatomic

ii) – massless string (loaded)




uniform tension $T$

uniform $\begin{cases} \text{mass } m \\ \text{separation } \ell \end{cases}$

For i)

$$\frac{1}{2}k(x_{i+1}-x_i)^2$$

$$L = \sum_{i=1}^{N}\left(\frac{1}{2}m\dot{x}_i^2 - \left(\frac{1}{2}k(x_i - x_{i-1})^2 + \frac{1}{2}k(x_{i+1}-x_i)^2\right)\right)$$

$$\begin{cases} x_0 \equiv 0 \\ x_{N+1} \equiv 0 \end{cases}$$

or simply

$$L = \sum_{i=1}^{N} \left( \frac{1}{2} m \dot{x}_i^2 - \frac{k}{2} (x_{i+1} - x_i)^2 \right)$$

{ Compressional Modes

For ii.),

$$L = \sum_{i=1} \left( \frac{1}{2} m \dot{y}_i^2 - \frac{\tau}{2\ell} (y_{i+1} - y_i)^2 \right)$$

{ transverse modes

≈ identical systems.

∴ hereafter, focus on i.)

motivations for i.) — monatomic chain is simplest example of elastic wave in solid

— step toward continuous system
i.e. now discrete → masses separated by $\ell$

Proceeding:

$$m \ddot{x}_i - k \left[ (x_{i+1} - x_i) + (x_{i-1} - x_i) \right] = 0$$

$$\ddot{x}_i + \frac{k}{m} \left[ 2x_i - (x_{i+1} + x_{i-1}) \right] = 0$$

$$x_i = \hat{x}_i e^{-i\omega t}$$

$$\left(\frac{2k}{m} - \omega^2\right)\hat{x}_i - \frac{k}{m}\left(\hat{x}_{i-1} + \hat{x}_{i+1}\right) = 0$$

For eigenvalues, $\det \underline{\underline{A}} = 0$

$$\underline{\underline{A}} = \begin{vmatrix} \frac{2k}{m} - \omega^2 & -k/m & & & \\ -k/m & \frac{2k}{m} - \omega^2 & -k/m & 0 & \\ & -k/m & \frac{2k}{m} - \omega^2 & -k/m & \\ 0 & & -k/m & \frac{2k}{m} - \omega^2 & -k/m \end{vmatrix}$$

i.e. A tri-diagonal.

Now, taking masses separated by $\ell$, take

$$\hat{x}_n \sim e^{i(n\ell)\underset{\text{wave-vector}}{\alpha}} \qquad \begin{cases} n \equiv \text{bead \#} \\ \alpha \equiv \text{wave \#} \\ \ell \equiv \text{spacing} \end{cases}$$

$$\Rightarrow \left(\frac{2k}{m} - \omega^2\right)e^{i[i\ell\alpha]} - \frac{k}{m}\left(e^{i[(i+1)\ell\alpha]} + e^{i[(i-1)\ell\alpha]}\right) = 0$$

careful $i$'s.

$$\therefore \left(\frac{2k}{m} - \omega^2\right) - 2\frac{k}{m}\cos\left[(\alpha\ell)\right] = 0$$

Note: says $\hat{x}_{n+m} = e^{im\ell\alpha}\hat{x}_m$

phase displ $\sim m\ell$

Soln

$$\omega^2 = \frac{2k}{m}(2)\left[\frac{1-\cos(\alpha\ell)}{2}\right]$$

$$= \frac{4k}{m}\sin^2\left(\frac{\alpha\ell}{2}\right)$$

$$\Rightarrow \boxed{\omega^2 = \frac{4k}{m}\sin^2(\alpha\ell/2)}$$

$$\omega = 2\sqrt{k/m}\,|\sin\alpha\ell/2|$$

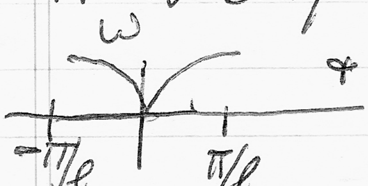
Note:

① − $\omega = \omega_{max}\,|\sin \ell\alpha/2|$ ; $\omega_{max}^2 = 4k/m$

$$\begin{cases} \omega(\alpha) = \omega(-\alpha) \\ \alpha' = \alpha + 2\pi/\ell \quad \text{leaves } \omega \text{ invariant} \end{cases}$$

i.e. need only define $\alpha$ on $[-\pi/\ell, \pi/\ell]$

i.e. First Brillouin Zone, only needed




& repeats

② − for $\ell\alpha/2 \ll 1$

i.e. wavelength $\alpha^{-1} \gg$ bead spacing $\ell$

$\Rightarrow$ continuum limit

then $\omega = \sqrt{k/m}\, \ell\, \alpha$

$= \alpha \left[ \ell \sqrt{k/m} \right]$

akin to acoustic wave

$\omega = k\, c_s$

$$\left\{ \begin{array}{l} k \leftrightarrow \alpha \\ c_s \leftrightarrow \ell\sqrt{k/m} \\ \frac{\gamma p}{\rho} \leftrightarrow \frac{\ell^2 k}{m} \end{array} \right.$$

$\ell^2 k \rightarrow$ stored elastic energy (springiness)

$m \rightarrow$ inertia

(3) – observe maximum frequency propagated is:

$\omega^2 = \omega_{max}^2 = 4k/m$ i.e. $\left\{ \begin{array}{ll} \omega^2 > \omega_{max}^2 & \text{not propagated} \\ \omega^2 < \omega_{max}^2 & \text{propagated} \end{array} \right.$

{ Chain acts as low-pass filter

{ Higher frequencies evanescent!

(4) – for propagation structure;

$\omega = 2\sqrt{k/m}\left[\sin(\alpha \ell/2)\right]$

$V_{gr} = d\omega/d\alpha = \ell\sqrt{k/m}\cos(\alpha\ell/2)$

i.e. $V_{gr} \cong \ell\sqrt{k/m} \sim c_{s\,eff}$ for $\alpha\ell \ll 1$

(a la' sound)

but $\lim_{\alpha \to \pi/\ell} v_{gr} = \ell\sqrt{k/m}\,\cos(\pi/2) \to 0$

i.e. modes at edge of Brillouin zone non-propagating

modes in middle of zone propagate at acoustic speed.

Can also observe that:

$$x_{i+1} + x_{i-1} - 2x_i = e^{i[\alpha\ell]i}\left(e^{i\alpha\ell} + e^{-i\alpha\ell} - 2\right)$$

$$= 2e^{i[i\alpha\ell]}\left(\cos\alpha\ell - 1\right)$$

so $\cos\alpha\ell / 1 \sim$ ratio of $(x_{i+1} + x_{i-1})/2x_i$

$\sim$ mean phase ratio

so $\underset{\text{(in zone)}}{\alpha\ell \ll 1} \Rightarrow$ neighbors on chain vibrate in phase $\to$ propagation
$\cos = 1$

$\underset{\text{(zone boundary)}}{\alpha\ell \sim \pi} \Rightarrow$ neighbors on chain vibrate out of phase $\to$ no propagation

$\cos = -1$

What if?:

→ Boundary Conditions

Can distinguish 2 cases: periodic B.C.'s; fixed end B.C.'s.

1) Periodic B.C.'s

Now, $x_i = Ae^{i[i\ell\alpha]}$

notational clarity ⇒ $x_n = Ae^{i[n\ell\alpha]}$

$$1 < n < N.$$

For periodic B.C.'s,

$$x_n = x_{n+N} \Rightarrow e^{iN\ell\alpha} = 1$$

↳ mode index.

$$\therefore \quad N\ell\alpha = 2\pi p$$

$$\Rightarrow \boxed{\alpha = \frac{2\pi p}{N\ell}} \qquad p = \begin{cases} 0, \pm 1, \ldots \pm \frac{1}{2}(N-1) & N \text{ odd} \\ 0, \pm 1, \ldots \frac{1}{2}N & N \text{ even} \end{cases}$$

Note: guarantees $N$ normal modes.

2) Fixed end B.C.'s: $X_0 = 0$, $X_{N+1} = 0$ } guarantees ends fixed

$\Rightarrow \quad X_0 = X_{N+1} = 0$

$$X_n = Ae^{in\alpha l} + Be^{-in\alpha l}$$

$$= A\sin(nl\alpha) + B\cos(nl\alpha)$$

$B = 0 \rightarrow n = 0$ ✓

$(N+1)\alpha l = p\pi \quad ; \quad p = 1, \ldots N$

mode index

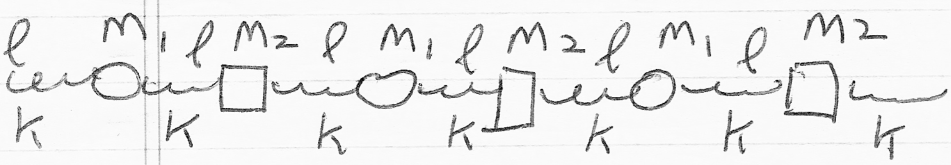
$\Rightarrow \quad \boxed{\alpha_p = p\pi / l(N+1)}$ { acts to quantize k

$\therefore \quad X_n(t) = A_n \sin\left(\frac{n\cancel{l}p\pi}{\cancel{l}(N+1)}\right) e^{-i\omega_p t}$

where $\omega_p^2 = \frac{4k}{m}\sin^2\left(\frac{p\pi\cancel{l}}{2\cancel{l}(N+1)}\right)$

- Diatomic Chain

→ consider slightly richer toy model, namely the diatomic chain




{ un-equal masses!

then, no loss of generality to associate

$m_1 \rightarrow x_{2n}$ (evens)
$m_2 \rightarrow x_{2n+1}$ (odds) } positions

∴ can immediately write dynamical equations

$$m_1 \ddot{x}_{2n} = -k\left(2x_{2n} - x_{2n-1} - x_{2n+1}\right)$$

$$m_2 \ddot{x}_{2n+1} = -k\left(2x_{2n+1} - x_{2n} - x_{2n+2}\right)$$

solution of form:

$$x_{2n} = A e^{i2n\ell\alpha} e^{-i\omega t} \qquad \text{(evens)}$$

$$x_{2n+1} = B e^{i(2n+1)\ell\alpha} e^{-i\omega t} \qquad \text{(odds)}$$

(consider one mass → 0, ∞)

$$-m_1 \omega^2 A = -k\left(2A - \left(e^{il\alpha} + e^{-il\alpha}\right) B\right)$$

$$-m_2 \omega^2 B = -k\left(2B - \left(e^{il\alpha} + e^{-il\alpha}\right) A\right)$$

$\Rightarrow$

$$(-m_1\omega^2 + 2k)A - k(2\cos l\alpha)\, B = 0$$

$$(-2k\cos l\alpha)A + (-m_2\omega^2 + 2k)B = 0$$

$\therefore$

$$\boxed{(\omega^2 - 2k/m_1)(\omega^2 - 2k/m_2) - \frac{4k^2}{m_1 m_2}\cos^2 l\alpha = 0}$$

$\Rightarrow$ dispersion relation:

$$\omega^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \pm k\left\{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^2 - \frac{4\sin^2(\alpha l)}{m_1 m_2}\right\}^{1/2}$$

$1/\mu \equiv 1/m_1 + 1/m_2$ $\Rightarrow$ reduced mass, as usual.

$$\omega^2 = k/\mu \pm k/\mu\left\{1 - \frac{4\mu^2}{m_1 m_2}\sin^2(\alpha l)\right\}^{1/2}$$

$\therefore$ dispersion relation:

$$\boxed{\omega^2 = \frac{k}{\mu}\left\{1 \pm 1\left\{1 - \frac{4\mu^2}{m_1 m_2}\sin^2(\alpha l)\right\}^{1/2}\right\}}$$

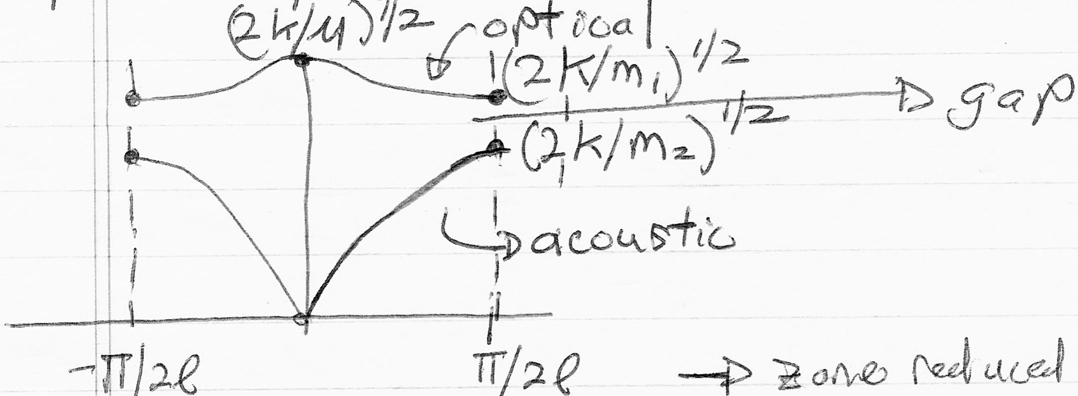
Can immediately observe:

→ system supports 2 modes

- low frequency → "acoustic" mode (ala' sound)
  → analogous to mode of monatomic chain

- high frequency → "optical" mode (ala' plasma) (vibration)
  → new

Can plot dispersion relation




$(m_2 > m_1)$

→ zone reduced by 2! (scale)

Note: - acoustic mode $\omega \sim \alpha \left( \frac{k\ell^2}{m_2 + m_1} \right)^{1/2}$

$\sim k c_{s_{eff}}$

as $k\ell \to 0$ ⇒ mass neighbors vibrate <u>in phase</u>

$x_n = x_{n+1}$

solid → phonon ($\omega = k c_s$)

optical mode $\omega \sim (2k/M)^{1/2}$

as $kl \to 0$; $m_1 x_n + m_2 x_{n+1} = 0$

i.e. neighboring masses vibrate <u>out</u> of phase, weighted by masses

solid → analogous collective mode is EM wave
$\omega^2 = \omega_{pe}^2 + c^2 k_1^2$ or plasmon
$\omega^2 = \omega_{pe}^2 + k^2 v_{Te}^2$

i.e. $k \to 0$, frequency constant!

→ Note gap → no propagation for

$$(2k/m_2)^{1/2} < \omega < (2k/m_1)^{1/2}$$

→ Consequence of fact
phonon ↔ inertia of heavy mass
optical ↔ inertia of light mass
(in $\omega_p^2$)

→ Transition to Continuum

To recover continuum $\left\{ \underline{\text{i.e.}} \begin{array}{l} \text{elastic medium} \\ \text{massive string} \end{array} \right.$

take $\begin{array}{l} N \to \infty \\ \ell \to 0 \end{array}$ $\underline{\underline{\text{with}}}$ $\underline{\underline{\text{constant}}}$ $L = (N+1)\ell$

$$\left\{ \begin{array}{l} \frac{m}{\ell} = \mu = \text{const.} \\ K\ell = \mathcal{K} = \text{const.} \end{array} \right.$$

Note: "$N \to \infty$" means $N > p$, for all modes $p$.

Then;

$$\omega_p^2 = \frac{4k}{m} \sin^2\left(\frac{p\pi}{2(N+1)}\right)$$

$$\cong \frac{4k}{m}\left(\frac{p\pi}{2(N+1)}\right)^2$$

$$= \frac{(p\pi)^2}{[(N+1)\ell]^2} \frac{k\ell^2}{m}$$

$$= \left(\frac{p\pi}{L}\right)^2 \left(\frac{\mathcal{K}}{\mu}\right) = \left(\frac{p\pi}{L}\right)^2 c_s^2$$

$$c_s^2 = k\ell^2/m = (k\ell)\,\ell/m = \mathcal{K}/\mu$$

$$\to \quad \omega^2 = k^2 c_s^2 \quad ; \quad \begin{array}{l} c_s^2 = \mathcal{K}/\mu \\ k = p\pi/L \end{array}$$